

# DEFORMATION SPACE OF A NON-UNIFORM 3-DIMENSIONAL REAL HYPERBOLIC LATTICE IN QUATERNIONIC HYPERBOLIC PLANE

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**ABSTRACT.** In this note, we study deformations of a non-uniform real hyperbolic lattice in quaternionic hyperbolic spaces. Specially we show that the representations of the fundamental group of the figure eight knot complement into  $PU(2, 1)$ , cannot be deformed in  $PSp(2, 1)$  out of  $PU(2, 1)$  up to conjugacy.

## 1. INTRODUCTION

In 1960's, A. Weil [15] proved a local rigidity of a uniform lattice  $\Gamma \subset G$  inside  $G$ , i.e., he showed that  $H^1(\Gamma, \mathfrak{g}) = 0$  for any semisimple Lie group  $G$  not locally isomorphic to  $SL(2, \mathbb{R})$ . This result implies that the canonical inclusion map  $i : \Gamma \hookrightarrow G$  is locally rigid up to conjugacy. In other words, for any local deformation  $\rho_t : \Gamma \rightarrow G$  such that  $\rho_0 = i$ , there exists a continuous family  $g_t \in G$  such that  $\rho_t = g_t \rho_0 g_t^{-1}$ . Weil's idea is further explored by many others but notably by Raghunathan [12] and Matsushima-Murakami [10]. Much later Goldman and Millson [4] considered the embedding of a uniform lattice  $\Gamma$  of  $SU(n, 1)$

$$\Gamma \hookrightarrow SU(n, 1) \hookrightarrow SU(m, 1), \quad m > n$$

and proved that there is still a local rigidity inside  $SU(m, 1)$  even if one enlarges the target group. More recently further examples of the local rigidity of a complex hyperbolic lattice in quaternionic Kähler manifolds are found in [5]

$$\Gamma \hookrightarrow SU(n, 1) \subset Sp(n, 1) \subset SU(2n, 2) \subset SO(4n, 4).$$

But all these examples deal with the standard inclusion map  $\Gamma \hookrightarrow G'$  to use the Weil's original idea about  $L^2$ -group cohomology. It has not been much studied yet when  $\rho : \Gamma \rightarrow G'$  is an arbitrary representation which is not an inclusion. In this note, we study deformations of a non-inclusion

<sup>1</sup>2000 *Mathematics Subject Classification.* 51M10, 57S25.

<sup>2</sup>*Key words and phrases.* Quaternionic hyperbolic space, complex hyperbolic space, local rigidity, representation variety.

<sup>3</sup>The author gratefully acknowledges the partial support of KRF grant (0409-20060066) and a warm support of IHES during his stay.

representation  $\rho_0$  of a non-uniform lattice  $\Gamma$  of  $PSL(2, \mathbb{C})$  in semisimple Lie groups  $PU(2, 1)$  and  $PSp(2, 1)$

$$\Gamma \xrightarrow{\rho_0} PU(2, 1) \subset PSp(2, 1).$$

This is a sequel to the previous paper [6] where the deformation of the standard inclusion  $\Gamma \hookrightarrow SO(3, 1) \subset SO(4, 1) \hookrightarrow Sp(n, 1)$  is studied but techniques are quite different. In [6], we used the group cohomology to prove the local rigidity following the path of [15, 12]. Here we use explicit coordinates and matrix calculations to prove a kind of a local rigidity, namely representations into  $PU(2, 1)$  cannot be deformed into  $PSp(2, 1)$  nontrivially. We calculate the dimension of a representation variety of the fundamental group of the figure eight knot complement in complex and quaternionic hyperbolic 2-plane using Thurston's idea.

In [3], Falbel constructed a special Zariski dense discrete representation  $\rho_0$  in  $PU(2, 1)$  with purely parabolic holonomy for a peripheral group. If  $g_1, g_2$  are generators of the fundamental group of the figure eight knot complement, their images in  $SU(2, 1)$

$$G_1 = \begin{bmatrix} 1 & 1 & \frac{-1-i\sqrt{3}}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{-1-i\sqrt{3}}{2} & -1 & 1 \end{bmatrix}$$

give such a special representation in  $PU(2, 1)$ . Note that this representation is not faithful.

More precisely we prove:

**Theorem 1.1.** *Let  $M$  be a figure eight knot complement which can be made up of two ideal tetrahedra in the quaternionic hyperbolic plane  $H_{\mathbb{H}}^2$  glued up along faces properly. For the space of representations  $\rho : \pi_1(M) \rightarrow PSp(2, 1)$  which do not stabilize a quaternionic line, the representation variety in  $PSp(2, 1)$  around a discrete representation  $\rho_0$ , is of real 3 dimension up to conjugacy. The representation variety into  $PU(2, 1)$  around  $\rho_0$  is of dimension 3 up to conjugacy as well. The variety around the conjugacy class  $[\rho_0]$  is parameterized by the three angular invariants of faces of one ideal tetrahedron.*

As a corollary we obtain

**Corollary 1.2.** *Any representation from the fundamental group of the figure eight knot complement into  $PU(2, 1)$  near the discrete representation  $\rho_0$ , cannot be deformed in  $PSp(2, 1)$  out of  $PU(2, 1)$  up to conjugacy.*

**Acknowledgements** The author thanks an anonymous referee for correcting some errors of the first version of the paper and constructive suggestions.

## 2. PRELIMINARIES

**2.1. Different models of hyperbolic space.** The set  $\mathbb{H}$  of quaternions are  $\{x = x_1 + ix_2 + jx_3 + kx_4 | x_i \in \mathbb{R}\}$  with the multiplication law  $ij = k$ ,  $jk = i$ ,  $i^2 = j^2 = k^2 = -1$ . We set  $\Im x = ix_2 + jx_3 + kx_4$  and  $\bar{x} = x_1 - ix_2 - jx_3 - kx_4$ . We call  $x$  pure imaginary if  $\bar{x} = -x$ . Quaternion number is a non-commutative division ring and by the abuse of notations, we will set  $x^{-1} = \frac{1}{x}$  to be the multiplicative inverse of  $x$ . Up to section 3, multiplication by quaternions on  $\mathbb{H}^n$  is **on the left** and matrices act **on the right**. Let  $J_0$  be

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Define  $\langle Z, W \rangle_0 = ZJ_0W^*$  where  $Z = (z_1, \dots, z_{n+1}) \in \mathbb{H}^{n+1}$ . Then  $A \in Sp(n, 1)$  if

$$AJ_0A^* = J_0.$$

Hence

$$A^{-1} = J_0A^*J_0.$$

One can define projective models, called **Siegel domains**, of the hyperbolic spaces  $H_{\mathbb{F}}^n$ ,  $\mathbb{F} = \mathbb{C}, \mathbb{H}$  as the set of negative lines in the Hermitian vector space  $\mathbb{F}^{n,1}$ , with Hermitian structure given by the indefinite  $(n, 1)$ -form

$$\langle Z, W \rangle_0 = ZJ_0W^*.$$

Namely  $H_{\mathbb{F}}^n$  is the left projectivization  $\mathbb{P}V_-$  of the set

$$V_- = \{Z \in \mathbb{F}^{n,1} : \langle Z, Z \rangle_0 < 0\}.$$

The boundary of the Siegel domain consists of projectivized zero vectors

$$V_0 = \{Z \in \mathbb{F}^{n,1} - \{0\} : \langle Z, Z \rangle_0 = 0\}$$

together with a distinguished point at infinity  $\infty$ . The finite points in the boundary carry the structure of the generalized Heisenberg group  $\mathbb{F}^{n-1} \times \mathfrak{SF}$  with the group law

$$(Z, t)(W, s) = (Z + W, t + s - 2\Im(\langle Z, W \rangle))$$

where  $\langle \langle Z, W \rangle \rangle = ZW^* = \sum z_i \bar{w}_i$  is the standard positive definite Hermitian form on  $\mathbb{F}^{n-1}$ . Motivated by this one can define **horospherical coordinates** for  $H_{\mathbb{F}}^n$

$$\{(z, t, v) \in \mathbb{F}^{n-1} \times \mathfrak{SF} \times \mathbb{R}_+\}.$$

From now on we will take  $n = 2$  so that we will deal with only two dimensional hyperbolic spaces. A coordinate change  $\psi$  from Heisenberg coordinates  $(z, t), z \in \mathbb{F}, t \in \mathfrak{SF}$  to the boundary of Siegel domain is

$$(1) \quad \left( \frac{-|z|^2 + t}{2}, z, 1 \right)$$

with one extra equation

$$\psi(\infty) = (1, 0, 0).$$

If  $U \in Sp(1)$ ,  $\mu \in Sp(1)$ ,  $r \in \mathbb{R}^+$ , the action fixing 0 and  $\infty$  is given by

$$(z, t) \rightarrow (r\mu^{-1}z\mu U, r^2\mu^{-1}t\mu).$$

See [8, 9]. In matrix form acting on the right

$$H_{\mu, U, r} = \begin{bmatrix} r\mu & 0 & 0 \\ 0 & \mu U & 0 \\ 0 & 0 & \frac{\mu}{r} \end{bmatrix}$$

So the hyperbolic isometry fixing  $\infty$  and 0 is determined by  $\mu, \nu = \mu U \in Sp(1)$  and  $r \in \mathbb{R}^+$ , so it is 7 dimensional. For complex hyperbolic space  $H_{\mathbb{C}}^2$ ,  $\mu = 1$  and  $U \in U(1)$ .

**Lemma 2.1.** *The set of isometries fixing three points on the ideal boundary of  $H_{\mathbb{H}}^2$ , which do not lie on the quaternionic line, is one dimensional whereas it is unique in  $H_{\mathbb{C}}^2$ .*

**Proof:** We may assume that three points are

$$\infty, 0, (1, it)$$

up to the action of  $Sp(2, 1)$ . If  $H_{\mu, U, r}$  fixes  $(1, it)$ , then  $U = 1, r = 1$  and  $\mu^{-1}it\mu = it$ . It is easy to show that  $\mu = e^{i\theta}$ , showing that  $\mu$  has one degree of freedom. ■

The Heisenberg group acts by right multiplication:

$$T_{(z,t)}(\zeta, v) = (z + \zeta, t + v - 2\Im\zeta\bar{z}).$$

In matrix form acting on  $\mathbb{H}^{2,1}$  on the right

$$T_{(z,t)} = \begin{bmatrix} 1 & 0 & 0 \\ -\bar{z} & 1 & 0 \\ \frac{-|z|^2+t}{2} & z & 1 \end{bmatrix}$$

Then a hyperbolic isometry fixing  $\infty$  and  $(z, t)$  is

$$T_{(-z,-t)} \circ H_{\mu, U, r} \circ T_{(z,t)} = \begin{bmatrix} r\mu & 0 & 0 \\ r\bar{z}\mu - \nu\bar{z} & \nu & 0 \\ r\frac{-|z|^2-t}{2}\mu + z\nu\bar{z} + \frac{\mu-|z|^2+t}{r} & -z\nu + \frac{\mu}{r}z & \frac{\mu}{r} \end{bmatrix}$$

where  $\nu = \mu U \in Sp(1)$ . This group is also 7 dimensional determined by  $\mu, \nu \in Sp(1)$ ,  $r \in \mathbb{R}^+$ . We call  $T_{(z,t)}$  a pure parabolic whereas  $H_{(\mu, U, 1)} \circ T_{(z,t)}$  ellipto-parabolic if it fixes a unique point at infinity.

**Lemma 2.2.** *Two pure parabolic elements  $T_{(z,t)}, T_{(w,s)}$  commute if  $w\bar{z} = z\bar{w}$ , i.e.  $w = rz$  for some real  $r$ .*

**Proof:** A direct calculation shows that two elements commute iff  $w\bar{z} = z\bar{w}$ , which implies that  $w\bar{z}$  is real. ■

There is one more isometry interchanging  $\infty$  and  $0$  whose matrix form is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and in Heisenberg coordinates

$$I(z, t) = \left( -\frac{2}{|z|^2 + \bar{t}} z, 4 \frac{\bar{t}}{|z|^4 + |t|^2} \right).$$

A reflection with respect to  $H_{\mathbb{R}}^2$  in  $H_{\mathbb{H}}^2$  is an isometry. In Heisenberg coordinates, it is

$$(z, t) \rightarrow (\bar{z}, \bar{t}).$$

So

$$(z, t) \rightarrow \left( \frac{-2}{|z|^2 + t} \bar{z}, \frac{4t}{|z|^4 + |t|^2} \right)$$

is an isometry interchanging  $\infty$  and  $0$ . Composing with  $(z, t) \rightarrow (r\mu z U \mu^{-1}, r^2 \mu t \mu^{-1})$  we get

$$R(z, t) = \left( r\mu \frac{-2}{|z|^2 + t} \bar{z} \nu, r^2 \mu \frac{4t}{|z|^4 + |t|^2} \mu^{-1} \right),$$

where  $r \in \mathbb{R}^+, \mu, \nu \in Sp(1)$ .

**2.2. Angular invariant.** To define angular invariant we introduce the **unit ball model**  $\{Z \in \mathbb{F}^n : \|Z\| < 1\}$  to make it compatible with existing literatures where  $\mathbb{F}^n$  is equipped with the standard positive definite Hermitian form. Two points  $(0', -1)$  and  $(0', 1)$  will play a special role. There is a natural map from a unit ball model to  $\mathbb{P}(\mathbb{F}^{n,1})$  where  $\mathbb{F}^{n,1}$  is equipped with a standard  $(n, 1)$  Hermitian product

$$\langle z, w \rangle = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n - z_{n+1} \bar{w}_{n+1},$$

defined as

$$(w', w_n) \rightarrow (w', w_n, 1).$$

From unit ball model to the horospherical model, one defines the coordinates change as

$$(z', z_n) \rightarrow \left( \frac{z'}{1 + z_n}, \frac{2\Im z_n}{|1 + z_n|^2}, \frac{1 - |z_n|^2 - |z'|^2}{|1 + z_n|^2} \right).$$

Its inverse from the horospherical model to  $\mathbb{P}\mathbb{F}^{n,1}$  is given by

$$(2) \quad (\xi, v, u) = \left[ \left( \xi, \frac{1 - |\xi|^2 - u + v}{2}, \frac{1 + |\xi|^2 + u - v}{2} \right) \right],$$

where  $v$  is pure imaginary, i.e.,  $iv$  in complex case, and  $iv_i + jv_2 + kv_3$  in quaternionic case. Note this coordinate change is different from (1) since we used a different Hermitian product. According to this coordinate change,  $(0', 1) = [(0', 1, 1)]$  corresponds to the identity element  $(0', 0)$  in Heisenberg group,  $(0', -1) = [(0', -1, 1)]$  to  $\infty$ , and  $(0', 0)$  to  $(0, 0, 1)$ .

**Definition 2.3.** The complex Cartan angular invariant  $\mathbb{A}(x_1, x_2, x_3)$  of the ordered triples  $(x_1, x_2, x_3)$  in  $\partial H_{\mathbb{C}}^n$  is introduced by Cartan [1] and defined to be the argument between  $-\frac{\pi}{2}$  and  $\frac{\pi}{2}$  of the Hermitian triple product

$$-\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle = -\langle \tilde{x}_1, \tilde{x}_2 \rangle \langle \tilde{x}_2, \tilde{x}_3 \rangle \langle \tilde{x}_3, \tilde{x}_1 \rangle \in \mathbb{C}$$

where  $\tilde{x}_i$  is a lift of  $x_i$  to  $\mathbb{C}^{n,1}$ . It can be obtained by, up to constant, integrating the Kähler form on  $H_{\mathbb{C}}^n$  over the geodesic triangle spanned by three points [2], hence it is a bounded cocycle. It satisfies the cocycle relation: for  $(x_1, x_2, x_3, x_4) \in \partial H_{\mathbb{C}}^n$

$$(3) \quad \mathbb{A}(x_1, x_2, x_3) + \mathbb{A}(x_1, x_3, x_4) = \mathbb{A}(x_1, x_2, x_4) + \mathbb{A}(x_2, x_3, x_4).$$

The quaternionic Cartan angular invariant of a triple  $x$ ,  $0 \leq \mathbb{A}_{\mathbb{H}}(x) \leq \pi/2$ , is the angle between the first coordinate line  $\mathbb{R}e_1 = (\mathbb{R}, 0, 0, 0) \subset \mathbb{R}^4$  and the Hermitian triple product

$$\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle = \langle \tilde{x}_1, \tilde{x}_2 \rangle \langle \tilde{x}_2, \tilde{x}_3 \rangle \langle \tilde{x}_3, \tilde{x}_1 \rangle \in \mathbb{H},$$

where we identify  $\mathbb{H}$  and  $\mathbb{R}^4$ .

Note that the invariant is unchanged under the homothety by nonzero real numbers, i.e., the triples  $x$  and  $rx$  have the same angular invariant.

**Proposition 2.4.** ([11],[7]). Let  $x = (x_1, x_2, x_3)$  and  $y = (y_1, y_2, y_3)$  be pairs of distinct triples of points in  $H_{\mathbb{H}}^n$ . Then  $\mathbb{A}_{\mathbb{H}}(x) = \mathbb{A}_{\mathbb{H}}(y)$  if and only if there is an isometry  $f \in PSp(n, 1)$  such that  $f(x_i) = y_i$  for  $i = 1, 2, 3$ .

**Proof:** Applying a homothety by nonzero real number, we may assume that our triples have Hermitian products  $X = \langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle$  and  $Y = \langle \tilde{y}_1, \tilde{y}_2, \tilde{y}_3 \rangle$  with  $|X| = |Y|$ .

Now let us assume that  $\mathbb{A}_{\mathbb{H}}(x) = \mathbb{A}_{\mathbb{H}}(y)$ . This and  $|X| = |Y|$  imply that there is an orthogonal transformation  $M \in SO(3) \times \{\text{id}\}$  acting on  $\mathbb{H} = \mathbb{R}^4$  that leaves invariant the real axis in  $\mathbb{H}$  and maps  $X$  to  $Y$ . Since the conjugation action of  $Sp(1)$  in  $\mathbb{H}$  is  $SO(3)$  action, there is  $\mu \in Sp(1)$  such that

$$\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle = \mu \langle \tilde{y}_1, \tilde{y}_2, \tilde{y}_3 \rangle \bar{\mu}.$$

To finish the proof it is enough to choose lifts  $\tilde{x}_i$  and  $\tilde{y}_i$  of points  $x_i$  and  $y_i$ ,  $i = 1, 2, 3$ , so that  $\langle \tilde{x}_i, \tilde{x}_j \rangle = \langle \tilde{y}_i, \tilde{y}_j \rangle$ . Indeed, then there is  $A \in Sp(n, 1)$  such that  $A(\tilde{x}_i) = \tilde{y}_i$ ,  $i = 1, 2, 3$ . Then it descends to an element  $f \in PSp(n, 1)$  such that  $f(x_i) = y_i$  for  $i = 1, 2, 3$ .

To obtain those lifts, we first replace  $\tilde{y}_1$  by  $\mu \tilde{y}_1$  (still denote it by  $\tilde{y}_1$ ) and get  $\langle \tilde{x}_1, \tilde{x}_2 \rangle \langle \tilde{x}_2, \tilde{x}_3 \rangle \langle \tilde{x}_3, \tilde{x}_1 \rangle = \langle \tilde{y}_1, \tilde{y}_2 \rangle \langle \tilde{y}_2, \tilde{y}_3 \rangle \langle \tilde{y}_3, \tilde{y}_1 \rangle$ . Replacing  $\tilde{x}_2$  and  $\tilde{x}_3$  by  $\mu_2 \tilde{x}_2$  and  $\mu_3 \tilde{x}_3$  if necessary, we can make  $\langle \tilde{x}_2, \tilde{x}_3 \rangle = \langle \tilde{y}_2, \tilde{y}_3 \rangle$  and  $\langle \tilde{x}_3, \tilde{x}_1 \rangle = \langle \tilde{y}_3, \tilde{y}_1 \rangle$ . Now the equation becomes

$$\langle \tilde{x}_1, \tilde{x}_2 \rangle \langle \tilde{x}_2, \tilde{x}_3 \rangle \langle \tilde{x}_3, \tilde{x}_1 \rangle = |\mu_2|^2 |\mu_3|^2 \langle \tilde{y}_1, \tilde{y}_2 \rangle \langle \tilde{y}_2, \tilde{y}_3 \rangle \langle \tilde{y}_3, \tilde{y}_1 \rangle,$$

and we get  $\langle \tilde{x}_1, \tilde{x}_2 \rangle = r \langle \tilde{y}_1, \tilde{y}_2 \rangle$  where  $r = |\mu_2||\mu_3|$ . Then replacing  $\tilde{x}_1$ ,  $\tilde{x}_2$ ,  $\tilde{x}_3$  and  $\tilde{y}_1$  by  $r^{-1} \tilde{x}_1$ ,  $r^{-1} \tilde{x}_2$ ,  $r \tilde{x}_3$  and  $r^2 \tilde{y}_1$  respectively, we finally get  $\langle \tilde{x}_i, \tilde{x}_j \rangle = \langle \tilde{y}_i, \tilde{y}_j \rangle$ , and hence a desired  $f \in PSp(n, 1)$ . ■

The converse is trivial. ■

**Theorem 2.5.** *For distinct points  $x_1, x_2, x_3 \in \partial H_{\mathbb{H}}^n$ , let  $\sigma_{12}$  and  $\Sigma_{12}$  be real and quaternionic geodesics containing the two points  $x_1$  and  $x_2$ , and  $\Pi: H_{\mathbb{H}}^n \rightarrow \Sigma_{12}$  be the orthogonal projection. Then*

$$|\tan \mathbb{A}_{\mathbb{H}}(x)| = \sinh(d(\Pi x_3, \sigma_{12}))$$

where  $d$  is the hyperbolic distance in  $H_{\mathbb{H}}^n$ .

**Proof:** Up to an isometry (in the unit ball model of  $H_{\mathbb{H}}^n$ ), we may assume that the triple  $x$  consists of  $x_1 = (0, -1)$ ,  $x_2 = (0, 1)$ ,  $x_3 = (z', z_n)$ , whose lifts are  $\tilde{x}_1 = (0, -1, 1)$ ,  $\tilde{x}_2 = (0, 1, 1)$ ,  $\tilde{x}_3 = (z', z_n, 1)$ . In this setting  $\sigma_{12} = \{(0, t) : t \in \mathbb{R}, |t| < 1\}$ ,  $\Sigma_{12} = \{(0, z) : z \in \mathbb{H}, |z| < 1\}$ , and  $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle = 2(\bar{z}_n - 1)(1 + z_n)$ . So we get

$$|\tan \mathbb{A}_{\mathbb{H}}(x)| = \frac{|2 \operatorname{Im}(z_n)|}{1 - |z_n|^2}.$$

On the other hand, we note that  $\Pi(x_3) = z_n$  and  $\Sigma_{12}$  has the Poincaré ball model geometry of  $H_{\mathbb{R}}^4$  with sectional curvature  $-1$ . Choose a hyperbolic two plane in  $\Sigma_{12}$  that contains the geodesic  $\sigma_{12}$  and  $z_n$ . This plane is a Poincaré disk with curvature  $-1$ , where we can write  $z_n = \operatorname{Re} z_n + i|\operatorname{Im} z_n|$ . Let  $d$  be the hyperbolic distance between the point  $z_n$  and the real axis in that Poincaré disk. Then a direct calculation shows that  $\sinh(d) = |2 \operatorname{Im}(z_n)|/(1 - |z_n|^2)$ . ■

### 3. Q-STRUCTURE

After the complete hyperbolic structure on the figure eight knot complement was first given in [13], W. Thurston describes a complete, finite volume hyperbolic structure on the complement of the figure eight knot in the 3-sphere in [14] by gluing two tetrahedra. Then, he shows how to deform it to non-complete structures. All these structures have holonomies, which are homomorphisms of the fundamental group  $\Gamma$  of the figure eight knot complement (a 2 generators and 1 relator group) to  $SO(3, 1)^0$ . In fact, the **character variety**  $\chi(\Gamma, SO(3, 1)^0) = \operatorname{Hom}(\Gamma, SO(3, 1)^0)/SO(3, 1)^0$  is a smooth 1-dimensional complex manifold near the conjugacy class of the holonomy of the complete hyperbolic structure.

Let  $\rho = \Gamma \rightarrow PSp(2, 1)$  be the holonomy of the complete hyperbolic structure, followed by the embedding  $SO(3, 1)^0 \rightarrow SO(4, 1)^0 = PSp(1, 1) \rightarrow PSp(2, 1)$ . In [6], it is shown that any parabolicity-preserving local deformation around  $\rho$  preserves a quaternionic line again. It is conjectured that any deformation around  $\rho_0$  preserves a quaternionic line.

In this note, we prove that the component, containing  $\rho_0$  (introduced in section 1) with purely parabolic holonomy for a peripheral group, of the space of representations, which do not stabilize a quaternionic line, of the fundamental group of the figure eight knot complement in  $PSp(2, 1)$  is actually conjugate into  $PU(2, 1)$ .

**3.1. The figure eight knot complement.** Consider the following complex. Glue together two tetrahedra by identifying faces pairwise according to the pattern indicated in Figure 1.

The resulting complex has two 3-cells, four faces, two edges and one vertex. According to W. Thurston, the complement of the vertex is homeomorphic to the complement  $M$  of the figure eight knot in the 3-sphere. Identify the two 3-cells of  $M$  with regular ideal tetrahedra in (compactified)  $H_{\mathbb{R}}^3$ . This defines a hyperbolic structure on  $M$ , and therefore a homomorphism  $\rho_0 : \Gamma = \pi_1(M) \rightarrow SO(3, 1)^0$ . Here is how W. Thurston deforms it. Up to isometry, an ideal tetrahedron in  $H_{\mathbb{R}}^3$  is characterized by one complex number. Identifying the 3-cells of  $M$  with arbitrary ideal hyperbolic tetrahedra defines a hyperbolic structure on the complement of the 2 skeleton, depending on two complex parameters. One has to make sure that the gluing maps are isometries which extend the hyperbolic structure across the faces. In order for the hyperbolic structure to extend across the two edges, two algebraic equations must be satisfied, but one of them turns out to follow from the other as we can see from Lemma 3.4. As a consequence, one obtains a (complex) one parameter family of hyperbolic structures on  $M$ .

We adapt this construction to obtain homomorphisms  $\Gamma \rightarrow PSp(2, 1)$ . For this, we first classify ideal tetrahedra in  $H_{\mathbb{H}}^2$ , then introduce the relevant geometric structure, baptised  $Q$ -structure, and describe the compatibility equations along edges.

**3.2. Ideal triangles and tetrahedra in  $H_{\mathbb{H}}^2$ .** The group  $PSp(2, 1)$  is not transitive on triples of points of  $\partial H_{\mathbb{H}}^2$ . By Proposition 4, a pair of triples are mapped to each other by an isometry iff they have the same angular invariant. By Theorem 2.5 the angular invariant of a triple vanishes if and only if all points sit in a (compactified) totally real totally geodesic plane. It takes value  $\pi/2$  if and only if all points sit in a (compactified) quaternionic line.

If three ideal points  $x_1, x_2$  and  $x_3 \in \partial H_{\mathbb{H}}^2$  do not belong to a quaternionic line, the set of isometries that fix them is one dimensional by Lemma 2.1. It follows that if two triangles have equal angular invariants different from  $\pi/2$ , there is a one dimensional family of isometries that sends one to the other. Since the boundary of  $H_{\mathbb{H}}^2$  is 7-dimensional, for each  $c \in [0, \pi/2)$ , the space of ideal tetrahedra  $(x_1, \dots, x_4)$  with  $A_{\mathbb{H}}(x_1, x_2, x_3) = c$  up to isometry has dimension 6. It follows that ideal tetrahedra up to isometry depend on 7 parameters.

**Definition 3.1.** Let  $A = \{x_1, \dots, x_k\}$ ,  $k \geq 3$ , be a disjoint collection of ideal points on the ideal boundary of a rank one symmetric space  $X$ . The geometric center of  $A$  in  $X$  is the barycenter of the associated measure  $\delta_A = \sum \delta_{x_i}$ . In more details, let

$$F(x) = \int_{\partial X} B_o(x, \xi) d\delta_A(\xi)$$

be a function defined on  $X$  where  $B_o$  is the Busemann function normalized that  $B_o(o, \xi) = 0$ . Then it is strictly convex and its value goes to  $\infty$  as  $x$  tends to  $\partial X$ . The barycenter  $x_0$  of  $\delta_A$  can be written as

$$dF_{(x_0)}(\cdot) = \int_{\partial X} (dB_0)_{(x_0, \xi)}(\cdot) d\delta_A(\xi) = 0.$$

Let  $\Delta$  denote a fixed regular ideal tetrahedron in  $H_{\mathbb{R}}^3$ , let  $\dot{\Delta} \subset \Delta$  be the complement of the 1-skeleton. Given an ideal tetrahedron (i.e. 4 distinct points at infinity  $(x_1, \dots, x_4)$ ) in  $H_{\mathbb{H}}^2$ , the *straight singular simplex* spanning them is the continuous map of  $\dot{\Delta}$  to  $H_{\mathbb{H}}^2$  defined as follows. For each face  $(s_i, s_j, s_k)$  of  $\Delta$ , map the barycenter  $s_{ijk}$  of  $(s_i, s_j, s_k)$  in  $H_{\mathbb{R}}^3$  to the geometric barycenter  $x_{ijk}$  of  $(x_i, x_j, x_k)$  in  $H_{\mathbb{H}}^2$ . Map the orthogonal projection of  $s_{ijk}$  to the edge  $[s_i, s_k]$  to the orthogonal projection of  $x_{ijk}$  to the geodesic  $[x_i, x_j]$  defined by  $x_i$  and  $x_j$ , extend to an isometric map of edge  $[s_i, s_k]$  onto geodesic  $[x_i, x_j]$ . Then map each geodesic segment joining  $s_{ijk}$  to  $[s_i, s_j]$  to a constant speed geodesic segment joining  $x_{ijk}$  to the corresponding point  $[x_i, x_j]$ . Finally, map each geodesic segment joining the barycenter of  $(s_1, \dots, s_4)$  to a point on a face to a constant speed geodesic segment from the barycenter of  $(x_1, \dots, x_4)$  to the corresponding point in the previously defined parametrizations of faces. The obtained map being in general discontinuous along edges, but let us ignore edges.

### 3.3. $Q$ -structures.

**Definition 3.2.** Let  $M$  be a manifold. A  $Q$ -structure on  $M$  is an atlas of charts  $\phi_j = U_j \rightarrow H_{\mathbb{H}}^2$  which are continuous maps from open sets of  $M$  to  $H_{\mathbb{H}}^2$ , such that on  $U_j \cap U_k$ ,  $\phi_k = \psi_{jk} \circ \phi_j$  for some unique  $\psi_{jk} \in Sp(2, 1)$ .

Pick a pair of ideal tetrahedra whose faces have pairwise equal angular invariants, all different from  $\pi/2$ . Map the 3-cells of the figure eight knot complement  $M$  to  $H_{\mathbb{H}}^2$  using the straight singular simplices spanning chosen ideal tetrahedra. This defines a  $Q$ -structure on  $M$  with 2-skeleton deleted.

The  $Q$ -structure extends across faces. Indeed, each face  $(x_i, x_j, x_k)$  of tetrahedron  $T$  is isometric to a unique face  $(y_{i'}, y_{j'}, y_{k'})$  of tetrahedron  $T'$ . Let  $\psi \in Sp(2, 1)$  be the unique isometry which maps one face to the other. Then the straight singular simplices spanning  $T$  and  $\psi^{-1}(T')$  take the same values along the common face, and so form a chart defined in a neighborhood of that face. The elements of  $Sp(2, 1)$  realizing the change of charts with the previously defined two charts are identity and  $\psi$  respectively.

The  $Q$ -structure extends across an edge  $[s_i, s_j]$  if and only if its holonomy, an element of  $Sp(2, 1)$ , around that edge, equals the identity. Let us compute holonomy based at a point of  $T$ . A priori, we know that holonomy maps the image geodesic  $[x_i, x_j]$  to itself. As in section 2, stabilizer of  $[x_i, x_j]$  is  $\mathbb{R} \times Sp(1)Sp(1)$ . Therefore, vanishing of holonomy amounts to 7 equations. Since there are two edges, we get 14 equations. But we expect equations provided by the two edges to be dependent, as it happens in  $SO(3, 1)$ .

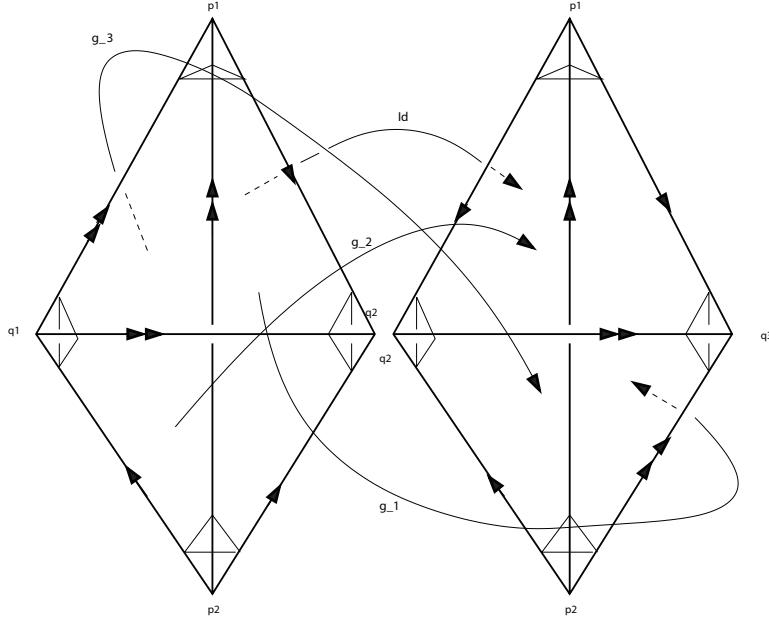


FIGURE 1. Gluing pattern of figure eight knot complement

**Proposition 3.3.** *Two holonomies around two edges in the complement of figure eight knot complement obtained by gluing two ideal tetrahedra are  $g_1^{-1}g_3g_2^{-1}g_1g_3^{-1}$  and  $g_2^{-1}g_3g_2g_1^{-1}$  where  $g_1, g_2, g_3$  are elements in  $Sp(2, 1)$  appearing in gluing pattern. One can permute the order of elements appearing in the products of  $g_1, g_2, g_3$ .*

**Proof:** By the gluing pattern, we obtain two pictures around two edges and the proof follows. See Figure 1. ■

**Lemma 3.4.** *If the holonomies around two edges multiply to be a pure parabolic element fixing a common point of two edges, which is automatically satisfied if the holonomies around edges are trivial, then one holonomy is determined by the other.*

**Proof:** Two tetrahedra glue up together to produce two edges  $p_1p_2$  and  $p_1q_2$ . Once we fix two tetrahedra, two holonomy  $H_1$  and  $H_2$  around  $p_1p_2$  and  $p_1q_2$  are hyperbolic isometries stabilizing them. But since two edges share  $\infty$ , product  $H_1H_2$  should fix  $\infty$ . Now put a restriction that  $H_1H_2$  is a pure parabolic isometry. Then

$$H_1 = \begin{bmatrix} q\alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \frac{\alpha}{q} \end{bmatrix}$$

and

$$H_2 = \begin{bmatrix} r\mu & 0 & 0 \\ r\bar{z}\mu - \nu\bar{z} & \nu & 0 \\ r\frac{-|z|^2-t}{2}\mu + z\nu\bar{z} + \frac{\mu-|z|^2+t}{r} & -z\nu + \frac{\mu}{r}z & \frac{\mu}{r} \end{bmatrix}$$

where  $q, r \in \mathbb{R}^+, \alpha, \beta, \mu, \nu \in Sp(1)$ . Then since  $H_1 H_2$  is a pure parabolic element fixing  $\infty$ ,

$$H_1 H_2 = T_{(w,s)} = \begin{bmatrix} 1 & 0 & 0 \\ -\bar{w} & 1 & 0 \\ \frac{-|w|^2+s}{2} & w & 1 \end{bmatrix}$$

for some  $(w, s)$ . From this equation we obtain

$$qr\alpha\mu = 1, \beta\nu = 1, \frac{\alpha\mu}{qr} = 1.$$

So  $r = \frac{1}{q}, \mu = \frac{1}{\alpha}, \nu = \frac{1}{\beta}$ . This shows that  $H_2$  is completely determined by  $H_1$ .  $\blacksquare$

**Corollary 3.5.** *The dimension of the space of real hyperbolic structures near the complete one on the figure eight knot complement is 2.*

**Proof:** An ideal tetrahedron in a quaternionic line (which is isometric to  $H_{\mathbb{R}}^4$ ) is determined by one complex variable. So two tetrahedra have two complex parameters. A holonomy around an edge belongs to the stabilizer of a geodesic, in our notation,  $H_{\mu, I, r}$  so that  $\mu \in U(1) \subset Sp(1)$ . So two holonomies around edges fix  $\infty$  in common, so its product is naturally a parabolic element. Then by Lemma 3.4, one holonomy determines the other. Hence there are two complex parameters with one complex equation and the solution space is of one complex dimension.  $\blacksquare$

First we calculate the dimension of the representation variety near  $\rho_0$  in  $PU(2, 1)$ .

**Proposition 3.6.** *The dimension of the component of the representation variety containing  $\rho_0$  from the fundamental group of the figure 8 knot complement to  $PU(2, 1)$  is 3 up to conjugacy.*

**Proof:** We claim that to choose an ideal tetrahedron there is 4 degrees of freedom. Choosing three points up to the action of  $PU(2, 1)$  is one degree of freedom corresponding to the Cartan angular invariant. Once three points are fixed, there are 3 degrees of freedom for the last vertex since the boundary of  $H_{\mathbb{C}}^2$  is three dimensional. Hence there are total 4 degrees of freedom to determine an ideal tetrahedron. To determine the second one, we claim that there is only one degree of freedom. By gluing pattern, three vertices of the second tetrahedron is determined according to the angular invariant. The last vertex of the second tetrahedron is connected to these three vertices to form 3 faces whose angular invariants are pre-determined by the gluing pattern. Since one angular invariant is determined if the other three are known in a tetrahedron by cocycle relation (3), two more angular invariants

will determine all the angular invariant. Since the last vertex can move around 3-dimensional space  $\partial H_{\mathbb{C}}^2$  with pre-determined two angular invariants there is only  $3 - 2$  degree of freedom to choose the second tetrahedron. So there are total  $4+1$  degrees of freedom to choose two tetrahedra to glue them according to the pattern. Then 5 points of two tetrahedra can be written as

$$p_1 = \infty, p_2 = 0, q_1 = (1, t), q_2 = (z, s), q_3 = (w, r)$$

where  $z, w \in \mathbb{C}, t, s, r \in \mathbb{S}\mathbb{C}$ . A coordinate change from horospherical coordinates  $(z, t), z \in \mathbb{C}, t \in \mathbb{S}\mathbb{C}$  to  $\mathbb{C}^{2,1}$  is

$$\left( \frac{-|z|^2 + t}{2}, z, 1 \right).$$

Then

$$\begin{aligned} p_1 &= (1, 0, 0), p_2 = (0, 0, 1), q_1 = \left( \frac{-1+t}{2}, 1, 1 \right), q_2 = \left( \frac{-|z|^2+s}{2}, z, 1 \right), \\ q_3 &= \left( \frac{-|w|^2+r}{2}, w, 1 \right). \end{aligned}$$

As above since there are 5 parameters, there is only one degree of freedom for  $(w, r)$ . This can be easily seen as follows. Note that the isometries in  $PU(2, 1)$

$$\begin{aligned} g_1 : (q_2, q_1, p_1) &\rightarrow (q_3, p_2, p_1) \\ g_2 : (p_2, q_1, q_2) &\rightarrow (p_1, q_2, q_3) \\ g_3 : (q_1, p_2, p_1) &\rightarrow (q_2, p_2, q_3) \end{aligned}$$

are all uniquely determined by their angular invariants. Hence  $g_1$  gives rise to one real equation in  $t, z, s, w, r$  which can be derived from the angular invariant of the faces  $(q_2, q_1, p_1)$  and  $(q_3, p_2, p_1)$ . Indeed one can do explicit calculations. Using the coordinate change formula in (2),

$$\begin{aligned} p_1 &= (0, -1, 1), p_2 = (0, 1, 1), q_1 = \left( 1, \frac{t}{2}, \frac{2-t}{2} \right), \\ q_2 &= \left( z, \frac{1-|z|^2+s}{2}, \frac{1+|z|^2-s}{2} \right), q_3 = \left( w, \frac{1-|w|^2+r}{2}, \frac{1+|w|^2-r}{2} \right) \end{aligned}$$

in  $\mathbb{C}^{2,1}$  with the standard  $(2, 1)$  Hermitian form  $\langle Z, W \rangle = z_1\bar{w}_1 + z_2\bar{w}_2 - z_3\bar{w}_3$  so that

$$\begin{aligned} -\langle q_2, q_1 \rangle \langle q_1, p_1 \rangle \langle p_1, q_2 \rangle &= \frac{|z-1|^2 - s - z + \bar{z} + t}{2} \\ -\langle q_3, p_2 \rangle \langle p_2, p_1 \rangle \langle p_1, q_3 \rangle &= |w|^2 - r. \end{aligned}$$

From  $\mathbb{A}(q_2, q_1, p_1) = \mathbb{A}(q_3, p_2, p_1)$ , we get

$$(4) \quad \frac{r}{|w|^2} = \frac{s + z - \bar{z} - t}{|z-1|^2}.$$

If  $A$  is a matrix representing  $g_1$ , since  $A \in U(2, 1)$ ,

$$AJ_0A^* = J_0.$$

From the fact that  $g_1$  fixes  $p_1$  and sends  $q_1$  to  $p_2$ , it is easy to show that  $A$  is of the form

$$\begin{bmatrix} a & 0 & 0 \\ a & b & 0 \\ a(-1-t)/2 & -b & \bar{a}^{-1} \end{bmatrix}$$

where  $|b| = 1$ . The fact that  $g_1$  sends  $q_2$  to  $q_3$  implies that

$$\begin{aligned} w &= bz\bar{a} - b\bar{a} = (z-1)b\bar{a}. \\ \frac{-|w|^2 + r}{2} &= \frac{-|z|^2|a|^2 + s|a|^2}{2} + z|a|^2 + \frac{-|a|^2 - t|a|^2}{2} \\ &= \frac{-|z-1|^2|a|^2 + s|a|^2 + z|a|^2 - \bar{z}|a|^2 - t|a|^2}{2}. \end{aligned}$$

From the first equation we have  $a = \frac{b\bar{w}}{\bar{z}-1}$  so  $|a|^2 = \frac{|w|^2}{|z-1|^2}$ . Substituting this into the second equation gives

$$\frac{-|w|^2 + r}{2} = \frac{-|w|^2 + (s+z-\bar{z}-t)\frac{|w|^2}{|z-1|^2}}{2}.$$

But this equation is just the angular invariant identity (4).

The same is true for  $g_2$ . The equation coming from  $g_3$  follows from the equations from  $g_1$  and  $g_2$  since three angular invariants determine the rest in a tetrahedron. In conclusion, out of 7 parameters  $t, s, r, z, w$ , there are two angular invariant equations, which makes 5 dimensional space as expected.

Now we consider holonomy relations. Since there is a unique element sending three points to other three points in general position according to the angular invariant,  $g_1, g_2, g_3$  are uniquely determined in terms of  $t, z, s, w, r$ . Then the holonomy  $g_1^{-1}g_3g_2^{-1}g_1g_3^{-1} \in SU(2, 1)$  around the edge connecting  $\infty$  and 0 is of the form

$$\begin{bmatrix} q\alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha/q \end{bmatrix}$$

where  $\alpha$  is a unit complex number and  $\beta = \alpha^{-2}$ . Now  $q\alpha$  is a function in  $t, z, s, w, r$ . To get a representation one should have

$$(5) \quad q\alpha = f(t, z, s, w, r) + g(t, z, s, w, r)i = 1.$$

Since  $q\alpha = f(t, z, s, w, r) + g(t, z, s, w, r)i$  is a complex number, it adds two more real equations. So there are 5 parameters with 2 equations, which gives at most 3 dimensional solution space. Since the other holonomy equation for the second edge follows from the first by Lemma 3.4, the solution space is 3 dimensional. ■

Next we deal with  $PSp(2, 1)$ . Here we would like to show that there is no deformation of representations from the fundamental group of the figure eight knot complement into  $PU(2, 1)$ , to the ones into  $PSp(2, 1)$  out of  $PU(2, 1)$  up to conjugacy. We do this by calculating the dimension of the variety of representations near  $\rho_0$  in  $PSp(2, 1)$  is also 3.

First we begin with a lemma.

**Lemma 3.7.** *Let  $\rho_1, \rho_2 : \Gamma \rightarrow SU(2, 1)$  be two non-conjugate Zariski dense representations. Then they are not conjugate even in  $Sp(2, 1)$ .*

**Proof:** Let  $\rho_1(\alpha) = A_1, \rho_1(\beta) = A_2$  and  $\rho_2(\alpha) = B_1, \rho_2(\beta) = B_2$  for the generators  $\alpha$  and  $\beta$  of the fundamental group of the figure eight knot complement such that  $A_i, B_i \in SU(2, 1)$ . Suppose  $\rho_1$  and  $\rho_2$  are conjugate in  $Sp(2, 1)$ , i.e., there exist  $X, Y \in Sp(2, 1)$  such that

$$XA_1X^{-1} = B_1, \quad XA_2X^{-1} = B_2$$

where  $X = X_1 + X_2j$ ,  $X^{-1} = X_3 + X_4j$  and  $X_1, X_2, X_3, X_4$  are  $3 \times 3$  complex matrices. From  $XX^{-1} = I$  there is a relation

$$X_1X_3 - X_2\bar{X}_4 = I, \quad X_1X_4 + X_2\bar{X}_3 = 0.$$

Also conjugation relation gives

$$(6) \quad X_1A_1X_3 - X_2\bar{A}_1\bar{X}_4 = B_1, \quad X_1A_1X_4 + X_2\bar{A}_1\bar{X}_3 = 0,$$

$$(7) \quad X_1A_2X_3 - X_2\bar{A}_2\bar{X}_4 = B_2, \quad X_1A_2X_4 + X_2\bar{A}_2\bar{X}_3 = 0.$$

Since  $\rho_1$  and  $\rho_2$  are not conjugate in  $SU(2, 1)$ ,  $X_2 \neq 0$ .

If  $X_1 = 0$ ,  $X = X_2j$ ,  $X_2 \in SU(2, 1)$ . But

$$X_2jA_i(X_2j)^{-1} = X_2jA_i(-j)X_2^{-1} = X_2\bar{A}_iX_2^{-1}.$$

It is easy to show that  $X_2\bar{A}_iX_2^{-1} \notin SU(2, 1)$  for generic element  $A_i$ . Since  $\rho_i$  is Zariski dense, we may assume that  $X_2\bar{A}_iX_2^{-1} \notin SU(2, 1)$  by choosing generators  $\alpha$  and  $\beta$  properly.

Hence  $X_1 \neq 0 \neq X_2$ . Then it is easy to show that  $X_3 \neq 0 \neq X_4$ .

Since  $\rho_1$  is a Zariski dense representation, we can choose  $A_1$  and  $A_2$  arbitrarily independent by choosing a different generators. Then for  $X_1$  and  $X_2$ , there are at most 18 complex parameters (indeed from  $X_iJ_0X_i^* = J_0$  there are less parameters than 18), whereas from Equations (6) and (7) there are at least  $9 \times 4$  complex equations. This forces that there are no solutions. ■

**Proposition 3.8.** *The dimension of the component of the representation variety of representations from the fundamental group of the figure 8 knot complement to  $PSp(2, 1)$ , containing  $\rho_0$ , which cannot be conjugate into  $PSp(1, 1)$ , is 3 up to conjugacy.*

**Proof:** We claim that there are 10 parameters to choose two tetrahedra. Put one tetrahedron in a standard position

$$p_1 = \infty, p_2 = 0, q_1 = (1, ia), q_2 = (z, t).$$

There is one degree of freedom for  $q_1$  up to  $PSp(2, 1)$  (corresponding to Cartan angular invariant, or more concretely corresponding to  $a \in \mathbb{R}$ ). But there is one parameter family of isometries fixing  $p_1, p_2, q_1$  by Lemma 2.1. So there are 6 degrees of freedom for  $q_2$  (since the dimension of the boundary of  $H_{\mathbb{H}}^2$  is 7), which makes 7 degrees of freedom to choose the first tetrahedron.

Once the first tetrahedron is chosen, three vertices of the second tetrahedron are determined by its angular invariant according to the gluing pattern. To choose the last vertex for the second tetrahedron, it is connected to three vertices to form three different faces, so their angular invariants are already determined in the first tetrahedron by gluing pattern. Hence there are  $6 - 3$  degrees of freedom to choose the last vertex for the second tetrahedron. In conclusion there are total  $7 + 3$  degrees of freedom to choose two tetrahedra. Note that our parameters are written in terms of  $(1, ia), (z, t)$  for the first tetrahedron, and  $(1, ib), (w, s)$  for the second, so the parameters are in 10 dimensional subspace of  $\mathbb{R} \times \mathbb{R} \times \mathbb{H} \times \mathbb{H} \times \mathfrak{SH} \times \mathfrak{SH}$ .

By the previous Lemma 3.4, a holonomy  $H_1$  around an edge being identity gives 7 equations. So we have 10 variables with 7 real equations. Note that  $H_1$  is the product of elements in  $PSp(2, 1)$  which can be written in terms of  $(1, ia), (z, t), (1, ib), (w, s)$  but  $H_1$  can be written as above in terms of  $r \in \mathbb{R}^+, \mu, \nu \in Sp(1) \subset \mathbb{H}$ . So  $H_1 = id$  produces 7 independent equations in terms of  $(1, ia), (z, t), (1, ib), (w, s)$ . Since the dimension of the variety into  $PU(2, 1)$  is already 3 and two non-conjugate Zariski dense representations in  $PU(2, 1)$  cannot be conjugate by an element in  $PSp(2, 1)$  by Lemma 3.7, the dimension of the variety into  $PSp(2, 1)$  should be also 3. This shows that every representation in  $PSp(2, 1)$  around  $\rho_0$  is conjugate into  $PU(2, 1)$ . So there is no deformation. ■

We suspect that this component containing a discrete representation  $\rho_0$  with purely parabolic holonomy for a peripheral group in  $PU(2, 1)$  is disjoint from the component containing the holonomy representation of the complete real hyperbolic structure in  $PSp(1, 1)$  in the representation variety in  $PSp(2, 1)$ .

#### 4. PARAMETERS FOR THE CHARACTER VARIETY IN $PU(2, 1)$ NEAR $\rho_0$

We showed that the dimension of the character variety from the fundamental group  $\Gamma$  of the figure eight knot complement to  $PU(2, 1)$  near  $[\rho_0]$  is 3. In this section we parameterize this space using angular invariants. We use the notations of Proposition 3.6. To parameterize the two ideal tetrahedra, we used five points

$$p_1 = \infty, p_2 = 0, q_1 = (1, t), q_2 = (z, s), q_3 = (w, r)$$

where  $z, w \in \mathbb{C}, t, s, r \in \text{Im } \mathbb{C}$ . From  $\mathbb{A}(q_2, q_1, p_1) = \mathbb{A}(q_3, p_2, p_1)$ , we had

$$\frac{r}{|w|^2} = \frac{s + z - \bar{z} - t}{|z - 1|^2}.$$

A direct calculation shows that from  $\mathbb{A}(q_1, p_2, p_1) = \mathbb{A}(q_2, p_2, q_3)$  we have

$$\arg\left(\frac{1-t}{2}\right) = \arg\left(\frac{(|z|^2 - s)(|w|^2 + r)(|w - z|^2 - r - w\bar{z} + z\bar{w} + s)}{8}\right).$$

Hence there were 5 independent parameters out of  $t, s, z, w, r$  to parameterize two ideal tetrahedra according to the gluing pattern. Finally the holonomy equation around the edge  $(p_1, p_2)$  gave two more real equations, hence the

real dimension of the character variety around  $[\rho_0]$  is 3. Here we give these 3 parameters in terms of Cartan angular invariants.

**Proposition 4.1.** *The character variety  $\chi(\Gamma, PU(2, 1))$  around  $[\rho_0]$  is parameterized by three angular invariants  $\mathbb{A}(p_1, p_2, q_j)$ ,  $j = 1, 2, 3$ .*

**Proof:** Since  $\mathbb{A}(q_3, p_2, p_1) = \mathbb{A}(q_2, q_1, p_1)$ , knowing three angular invariants  $\mathbb{A}(p_1, p_2, q_j)$ ,  $j = 1, 2, 3$  will completely determine the angular invariants of the first tetrahedron by cocycle relation (3). The angular invariants  $\mathbb{A}(p_1, p_2, q_j)$ ,  $j = 1, 2, 3$  are functions of only  $t, z, s$ . Gluing maps  $g_1, g_2, g_3$  relates variable  $t, z, s$  to the variable  $w, r$ . Hence the holonomy map  $g_1^{-1}g_3g_2^{-1}g_1g_3^{-1}$  relates  $t, z, s$  variable to  $w, r$ . In other words, holonomy map does not create any relation among  $t, z, s$ . This shows that three angular invariants  $\mathbb{A}(p_1, p_2, q_j)$ ,  $j = 1, 2, 3$  are independent. Hence these three parameters are parametrization of the character variety around  $[\rho_0]$ . ■

In [3], it is shown that the coordinates of tetrahedra corresponding to  $\rho_0$  are

$$p_1 = \infty, p_2 = 0, q_1 = (1, \sqrt{3}), q_4 = \left(-\frac{-1 - i\sqrt{3}}{2}, \sqrt{3}\right), q_3 = \left(\frac{-1 + i\sqrt{3}}{2}, \sqrt{3}\right)$$

and

$$\mathbb{A}(p_1, p_2, q_j) = \frac{\pi}{3}, \quad j = 1, 2, 3.$$

Hence in our coordinates  $[\rho_0] = (\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$ .

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